N11/5/MATHL/HP3/ENG/TZ0/SE

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MATHEMATICS
HIGHER LEVEL
PAPER 3 – SERIES AND DIFFERENTIAL EQUATIONS

Friday 4 November 2011 (morning)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 5]*

Find $\lim_{x \to \frac{1}{2}} \left(\frac{\left(\frac{1}{4} - x^2 \right)}{\cot \pi x} \right)$.

2. *[Maximum mark: 5]*

(a) Show that $n! \geq 2^{n-1}$, for $n \geq 1$. *[2 marks]*

(b) Hence use the comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges or diverges. *[3 marks]*

3. *[Maximum mark: 11]*

Consider the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n \times 2^n}$.

(a) Find the radius of convergence of the series. *[7 marks]*

(b) Hence deduce the interval of convergence. *[4 marks]*

4. *[Maximum mark: 8]*

(a) Using the integral test, show that $\sum_{n=1}^{\infty} \frac{1}{4n^2+1}$ is convergent. *[4 marks]*

(b) (i) Show, by means of a diagram, that $\sum_{n=1}^{\infty} \frac{1}{4n^2+1} < \frac{1}{4 \times 1^2 + 1} + \int_1^{\infty} \frac{1}{4x^2+1} \mathrm{d}x$.

(ii) Hence find an upper bound for $\sum_{n=1}^{\infty} \frac{1}{4n^2+1}$. *[4 marks]*

5. *[Maximum mark: 16]*

(a) Given that $y = \ln\left(\frac{1+e^{-x}}{2}\right)$, show that $\frac{dy}{dx} = \frac{e^{-y}}{2} - 1$. *[5 marks]*

(b) Hence, by repeated differentiation of the above differential equation, find the Maclaurin series for y as far as the term in x^3, showing that two of the terms are zero. *[11 marks]*

6. *[Maximum mark: 15]*

The real and imaginary parts of a complex number $x + iy$ are related by the differential equation $(x+y)\frac{dy}{dx} + (x-y) = 0$.

By solving the differential equation, given that $y = \sqrt{3}$ when $x = 1$, show that the relationship between the modulus r and the argument θ of the complex number is $r = 2e^{\frac{\pi}{3}-\theta}$.